Linear Systems & Signals

- Time-invariant & Linear Systems (TILS)
- TILS under exponential excitation
- Exponential decomposition of signals
- Exponential Fourier series
- Frequency characteristics
- Truncated Fourier series
- Periodicity of the Fourier series
- Limiting behaviour of the Fourier series
- Fourier transform
- Summary
- Trigonometric Fourier series (+)

Time-invariant Linear Systems (TILS)

$$x(t) \rightarrow y(t) = \mathcal{R}\{x(t)\}$$

$$x(t) \longrightarrow \boxed{\textbf{TILS}} \longrightarrow y(t) = \mathcal{R}\{x(t)\}$$

Linear System

$$x_1(t) \rightarrow y_1(t)$$
$$x_2(t) \rightarrow y_2(t)$$
$$a_1 x_1(t) + a_2 x_2(t) \rightarrow$$
$$\rightarrow a_1 y_1(t) + a_2 y_2(t)$$

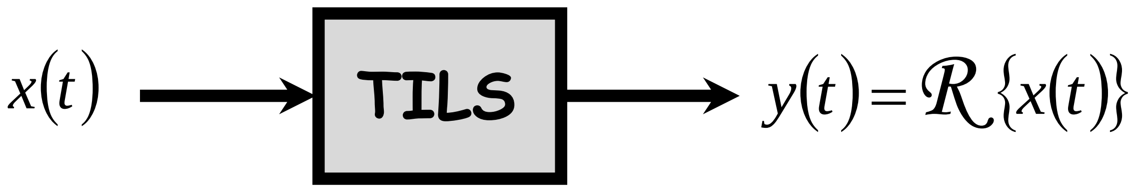
Time-invariant System

$$x(t) \rightarrow y(t)$$
$$x(t+\tau) \rightarrow y(t+\tau)$$

TILS

Time-invariant Linear Systems (TILS) - scaling

$$x(t) \rightarrow y(t) = \mathcal{R}\{x(t)\}$$



Linear System

$$x_1(t) \rightarrow y_1(t)$$
$$x_2(t) \rightarrow y_2(t)$$
$$a_1 x_1(t) + a_2 x_2(t) \rightarrow$$
$$\rightarrow a_1 y_1(t) + a_2 y_2(t)$$

Scaling

$$x_2(t) = 0$$
$$a_1 x_1(t) \rightarrow a_1 y_1(t)$$
$$a x(t) \rightarrow a y(t)$$

Time-invariant Linear Systems (TILS)

$$x(t) \rightarrow y(t) = \mathcal{R}\{x(t)\}$$

$$x(t) \longrightarrow \boxed{\textbf{TILS}} \longrightarrow y(t) = \mathcal{R}\{x(t)\}$$

Linear System

$$x_1(t) \rightarrow y_1(t)$$
$$x_2(t) \rightarrow y_2(t)$$
$$a_1x_1(t) + a_2x_2(t) \rightarrow$$
$$\rightarrow a_1y_1(t) + a_2y_2(t)$$

Time-invariant System

$$x(t) \rightarrow y(t)$$
$$x(t+\tau) \rightarrow y(t+\tau)$$

How to find output *y*(*t*) given input *x*(*t*)?

TILS under exponential excitation (special case)

$$x(t)=e^{st} \longrightarrow \boxed{\textbf{TILS}} \longrightarrow y(t)=?$$

$$x(t)=e^{st} \rightarrow y(t)=\mathbf{?}$$

Linear System

$$e^{st} \rightarrow y(t)$$
$$e^{s\tau}e^{st} \rightarrow e^{s\tau}y(t)$$

Time-invariant System

$$e^{st} \rightarrow y(t)$$
$$e^{s\tau}e^{st} = e^{s(t+\tau)} \rightarrow y(t+\tau)$$

$$y(t+\tau)=y(t)e^{s\tau}, s \in \boldsymbol{C}$$

Solution to the equation for *y*(*t*) ← *x*(*t*) = est:

- **by guess,**
- **by constructive proof.**

TILS under exponential excitation (by guess)

$$x(t)=e^{st} \longrightarrow \boxed{\text{TILS}} \longrightarrow \begin{array}{l} y(t)=H(s)e^{st} \\ y(t+\tau)=y(t)e^{s\tau} \end{array}$$

By guess - the solution to the equation

$y(t+\tau)=y(t)e^{s\tau}$ **is an exponential signal** $y(t)=H(s)e^{st}$

TILS changes the amplitude of the input exponential signal est by a factor *H*(*s*). Factor *H*(*s*) depends 1) on the exponent *s*$\in C$ and 2) the internal structure of the TILS (network of R, L, C elements).

Factor *H*(*s*) is also named a „transfer function” as:

$$H(s)=\frac{y(t)\leftarrow e^{st}}{e^{st}} = \frac{\textbf{response to the exponential excitation}}{\textbf{exponential excitation}}$$

TILS under exponential excitation
(constructive proof)

$$y(t+\tau)=y(t)e^{s\tau} \quad \Big| u=t+\tau, d(\cdot)/d\tau; du=d\tau$$

$$\frac{dy}{du}\times\underbrace{du/d\tau}_{=1}=s\underbrace{y(t)e^{s\tau}}_{=y(t+\tau)}$$

$$dy(u)/du=sy(u)$$

$$dy(u)/y(u)=sdu \qquad \Big|\int(\cdot)du$$

$$\ln y(u)=su+C$$

$$y(u)=e^{su}\underbrace{e^{C}}_{=H(s)}$$

$$y(t)=H(s)e^{st}$$

TILS fed by the exponential signal $x(t)=e^{st}$ responds with the exponential signal $y(t)=H(s)e^{st}$ where $H(s)$ is a „transfer function”.

Formula for the transfer function:

$$H(s)=\frac{y(t)\leftarrow e^{st}}{e^{st}} = \frac{\textbf{response to the exponential excitation}}{\textbf{exponential excitation}}$$

Transfer function depends on the structure of TILS (R, L, C elements) and the *s* value.

How to find TILS transfer function?

Formula for the transfer function:

$$H(s) = \frac{y(t) \leftarrow e^{st}}{e^{st}} = \frac{\textbf{response to the exponential excitation}}{\textbf{exponential excitation}}$$

Transfer function depends on the structure of TILS (network of R, L, C elements) and the *s* $\in \mathcal{C}$ value.

Generic approach to determine the transfer function *H*(*s*) is to find an ordinary differential equation (ODE) describing an RLC network (output *y*(*t*) in terms of input *x*(*t*)) and then „solving" the ODE using a substitution $x(t) = e^{st}$ oraz $y(t) = H(s)e^{st}$.

Engineering approach to determine the transfer function *H*(*s*) deploys the *„s"* (Laplace transform) approach.

Exponential decomposition of signals

$$e^{st} \longrightarrow \boxed{\text{TILS}} \longrightarrow y(t) = H(s)e^{st}$$

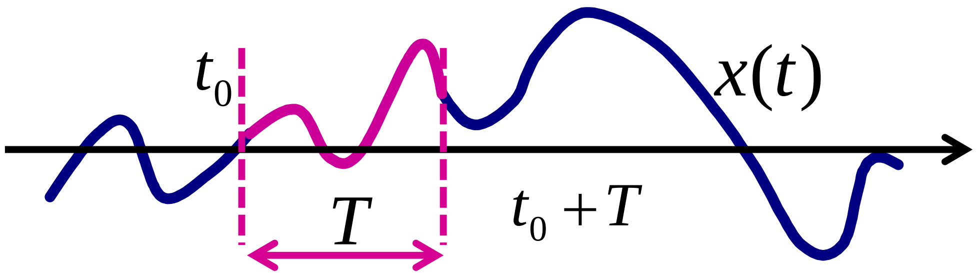
$$x(t) = \sum_n X_n e^{s_n t} \longrightarrow \boxed{\text{TILS}} \longrightarrow y(t) = \sum_n X_n H(s_n) e^{s_n t}$$

Decomposition of a signal $x(t)$ through exponential components $X_n\exp(s_n t)$ let us to determine a TILS response to any input signal $x(t) \rightarrow y(t)$ using TILS properties and provided the transfer function $H(s)$ is known.

Decomposition tools:

- **s = $\alpha + j\omega \in C$ – Laplace transform**
- **s = $j\omega \in I$ – Fourier series (transform)**

Exponential Fourier series



$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$t_0 \le t \le t_0 + T$$

$$\omega_0 = 2\pi/T = 2\pi f_0$$

T **– observation interval of a signal** $\boldsymbol{x(t)}$

ω_0 **= 2*π*/*T* – basic frequency [rad/s]**

f_0 **= 1/*T* =** ω_0**/2*π*– basic frequency [Hz]**

$$X_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt$$

Coefficients of exponential Fourier series

The exponential Fourier series represents a signal as a linear composition of complex harmonics exp(*j*$n\omega_0 t$) = cos($n\omega_0 t$) + *j* sin($n\omega_0 t$) with different amplitudes X_n.

Euler identity: exp(*j*$n\omega_0 t$) = cos($n\omega_0 t$) + *j* sin($n\omega_0 t$)

Exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$t_0 \le t \le t_0 + T, \quad \omega_0 = 2\pi/T$$

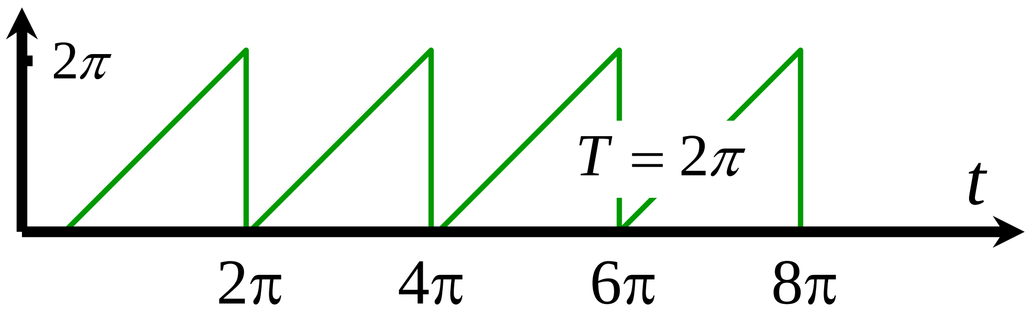
Composition of exponential terms

Power Maclaurin series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Composition of power terms

Triangle pulse train
Fourier series expansion



$$x(t) = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{n} e^{jnt}$$

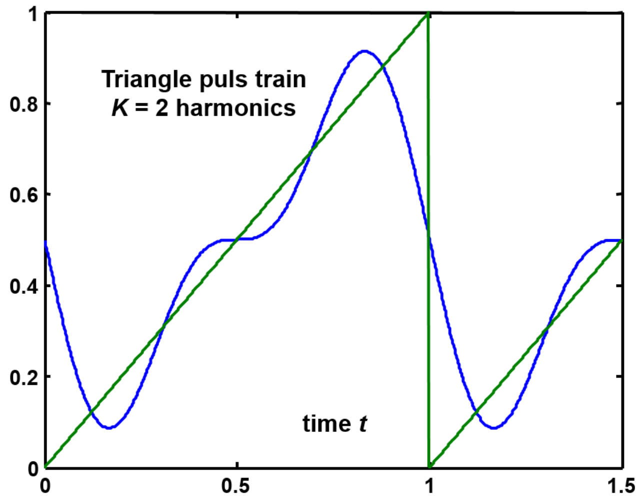
Indefinite integral:

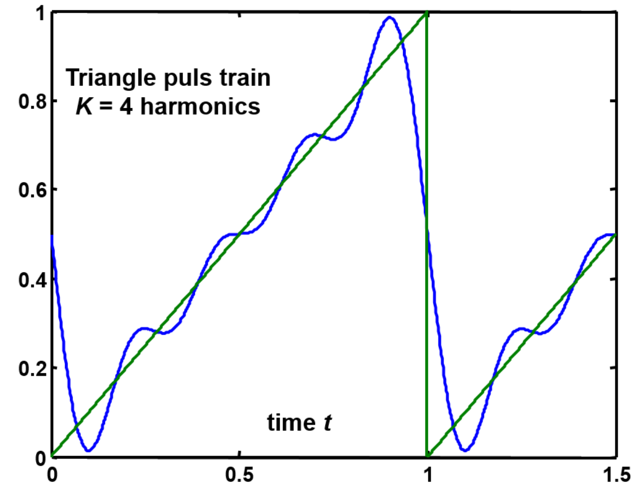
$$\int x e^{ax} dx = e^{ax}(ax-1)/a^2, \quad a \neq 0$$

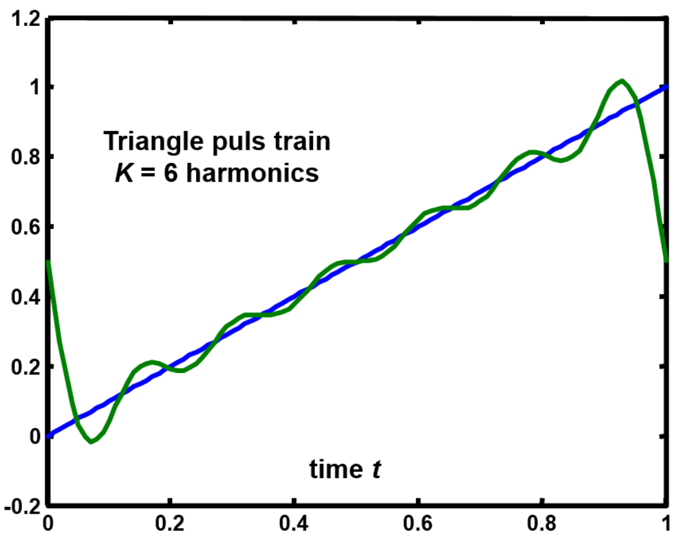
WORK IN PROGRESS

$$x(t) = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{n} e^{jnt} = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n} \sin(nt)$$

Triangle pulse train
Fourier series expansion

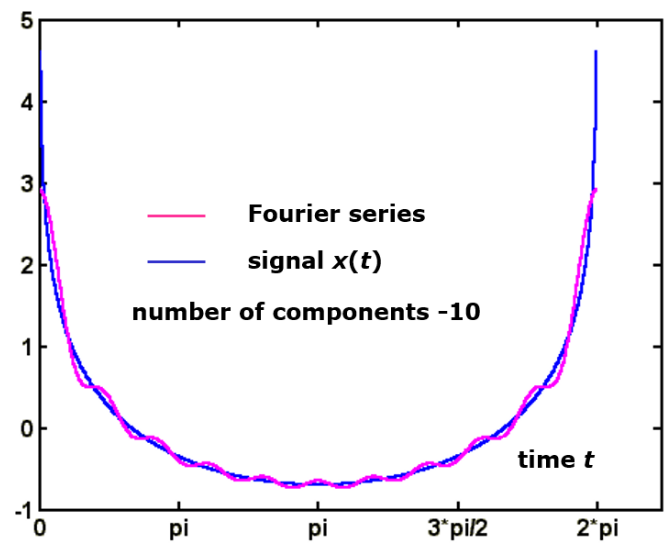






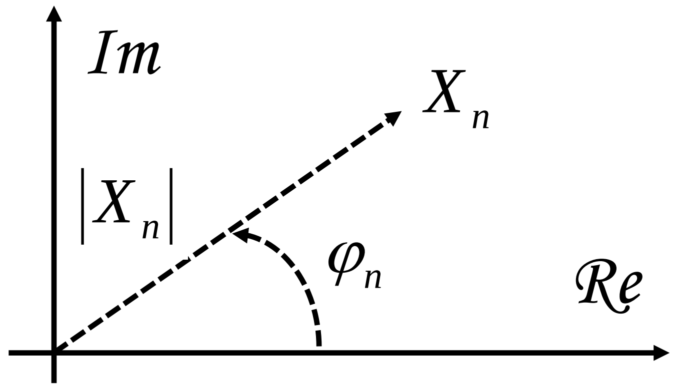
Remarkable Fourier series

$$x(t) = -\ln\left[2\sin(t/2)\right] = \sum_{n=1}^{\infty} \cos nt/n\,,\ 0 < t < 2\pi$$



Frequency characteristics

$$X_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt = |X_n|e^{j\varphi_n}$$



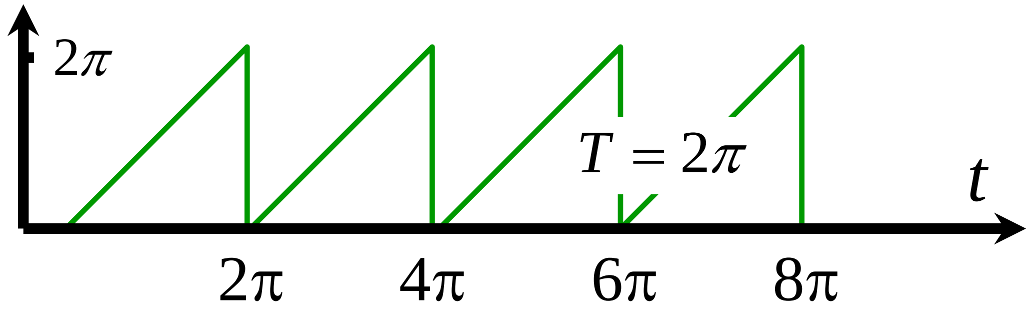
Amplitude-frequency (a-f) characteristic:

$$|X_n| = |X_n(n\omega_0)|, \quad n = 0, \pm 1, \pm 2 \ldots$$

Phase-frequency (p-f) characteristic:

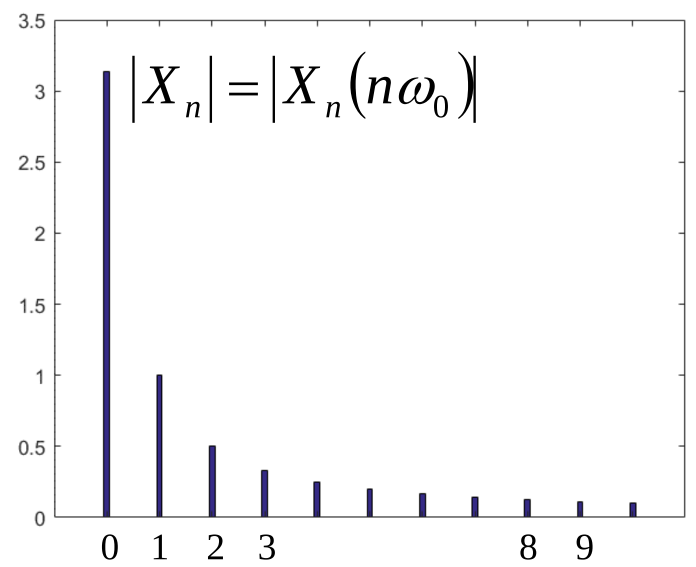
$$\varphi_n = \varphi_n(n\omega_0), \quad n = 0, \pm 1, \pm 2 \ldots$$

Triangle puls train
A-f and p-f characteristics



$$x(t) = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{n} e^{jnt} = \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|n|} e^{-j\,\mathrm{sgn}(n)\pi/2} e^{jnt}$$

$$|X_n| = \begin{cases} \pi, n = 0 \\ 1/|n|, n \neq 0 \end{cases} \qquad \varphi_n = \begin{cases} -\pi/2, n > 0 \\ +\pi/2, n < 0 \end{cases}$$



Truncated Fourier series

$$x(t) \approx x_K(t) = \sum_{n=-K}^{K} X_n e^{jn\omega_0 t}$$

$$t_0 \le t \le t_0 + T, \quad \omega_0 = 2\pi/T$$

$$X_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$

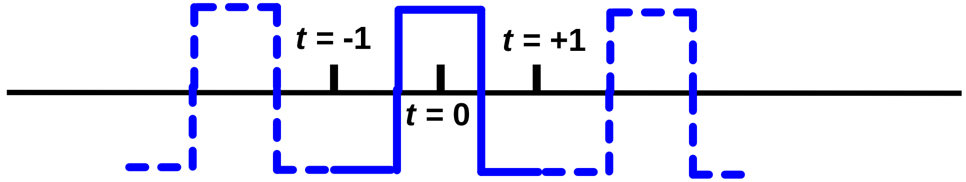
How long *K* = ? should the truncated Fourier series be?

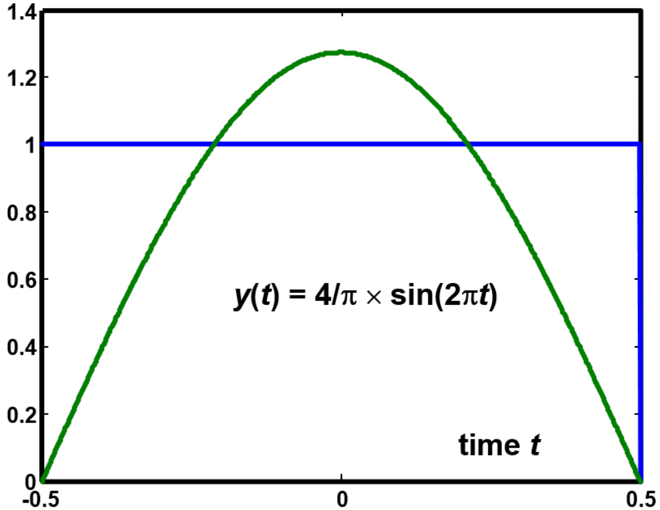
Basic approach – the number *K* of retained harmonics should provide a <u>relatively accurate approximation</u> of the original signal *x*(*t*).

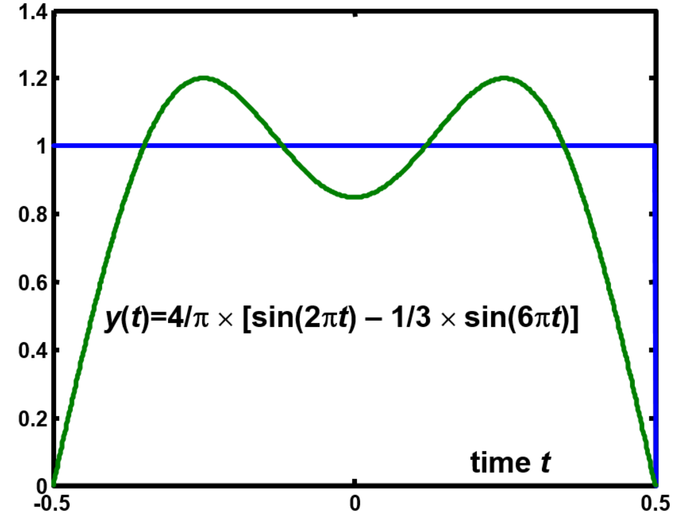
Relatively accurate approximation – what does it mean? Answers are provided by:

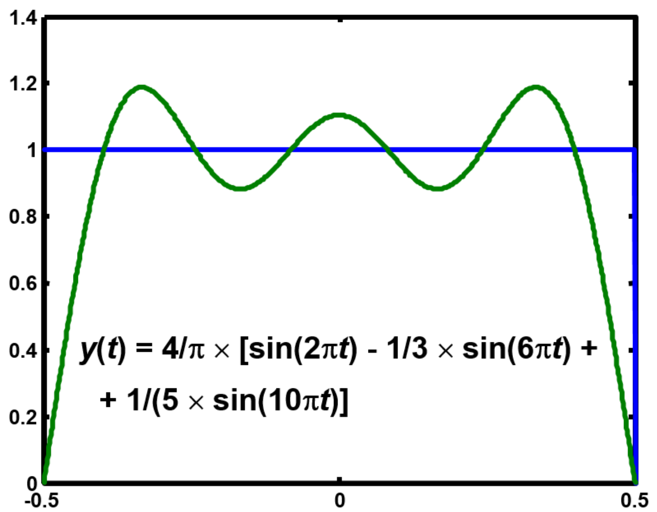
- **Parseval theorem,**
- **concept of fractional power.**

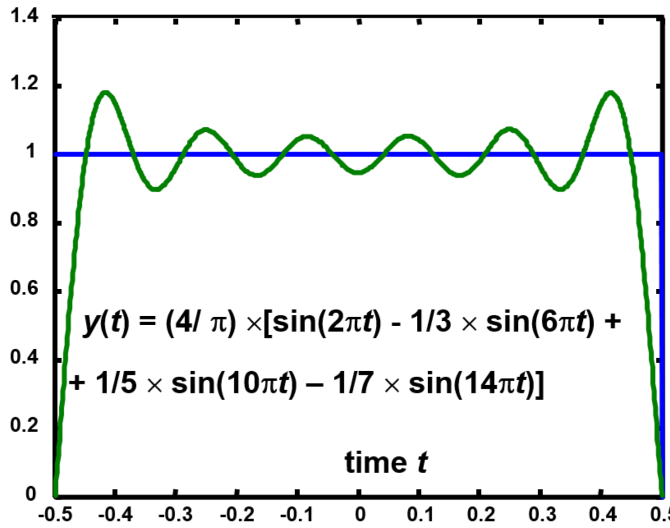
Rectangular puls train – Fourier series expansion











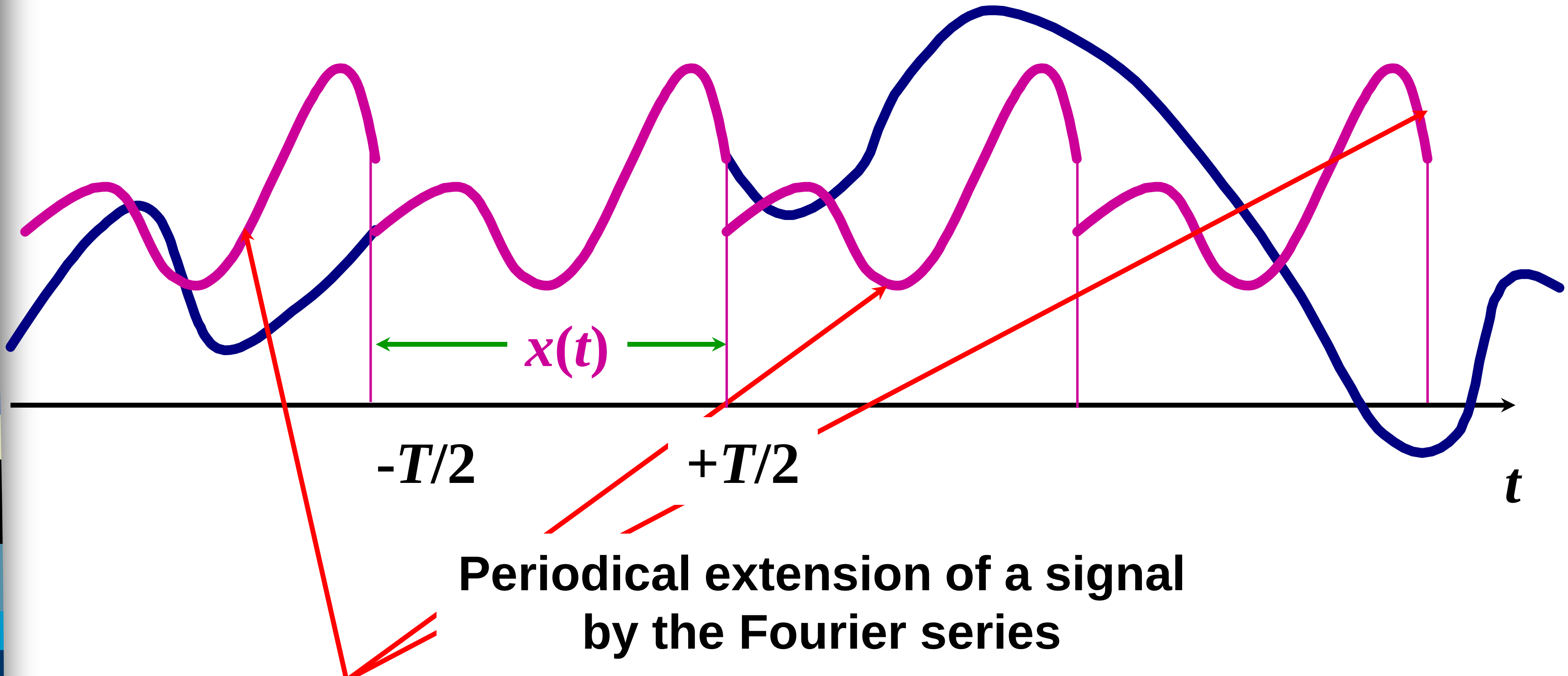
Periodicity of the Fourier series

$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, t_0 \le t \le t_0 + T$$

$$x(t+T)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 (t+T)}=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} e^{jn2\pi}=x(t)$$

Periodicity of the Fourier series generates a periodic extension of the *x*(*t*) signal beyond the $t_0 < t < t_0 + T$ interval.

Periodicity of the Fourier series

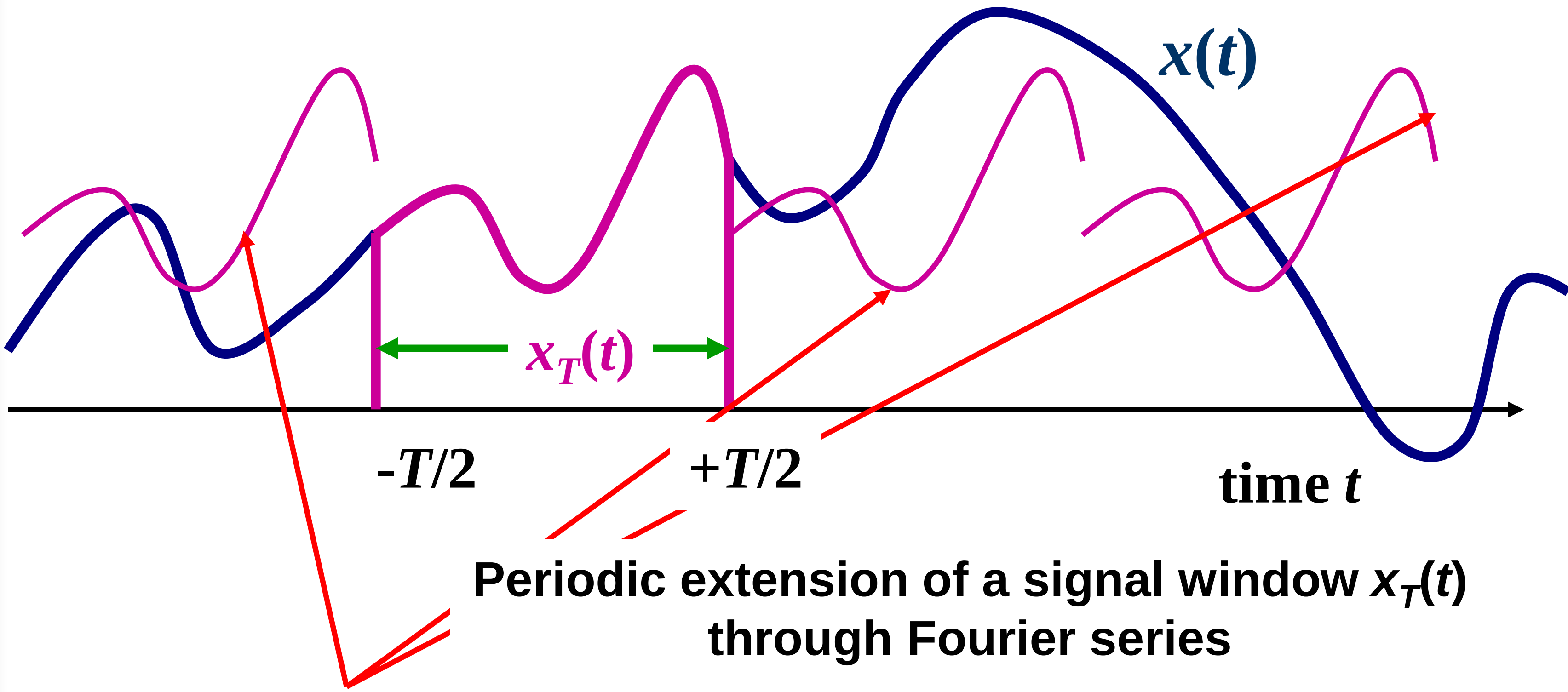


Periodical extension of a signal by the Fourier series

The Fourier series exactly fits a signal provided:
- **the signal is periodic and**
- **an expansion interval and a period are equal.**

Conclusion – Fourier series is suitable for periodic signal only.

Limiting behaviour of the Fourier series



Periodic extension of a signal window $x_T(t)$ through Fourier series

$$x_T(t) \xrightarrow[\text{Fourier series}]{T \to \infty} x(t)$$

Limiting behaviour of the Fourier series

$$x(t) \qquad \xrightarrow[n\omega_0 \to \omega]{T \to \infty,\ \omega_0 \to 0} \qquad x(t)$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

Fourier series $\longrightarrow$ Fourier transform pairs

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t} \longrightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$$X_n = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) e^{-jn\omega_0 t} dt \longrightarrow X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Fourier transform & notation

TRANSFORM

FORWARD

$$X(\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$X(\omega)=\mathcal{F}\{x(t)\}$$

INVERSE

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

$$x(t)=\mathcal{F}^{-1}\{X(\omega)\}$$

TRANSFORM PAIRS

$$x(t)\leftrightarrow X(\omega)$$

Fourier series, Fourier transform - condition for existence

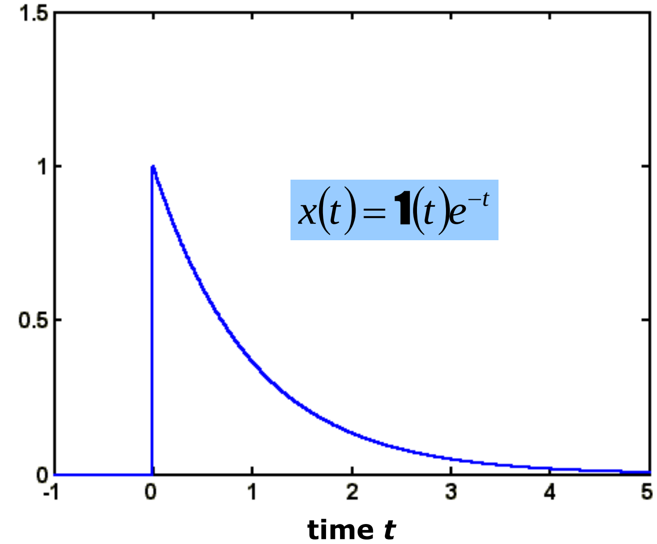
Within engineering applications it is commonly assumed that signals of <u>limited power *P*</u> can be expanded in a Fourier series (Fourier series is convergent):

$$P = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt < \infty$$

Within engineering applications it is commonly assumed that signals of limited energy *E* have a Fourier transform (improper Fourier integral exists):

$$E = \int_{-\infty}^{\infty} x^2(t) dt < \infty$$

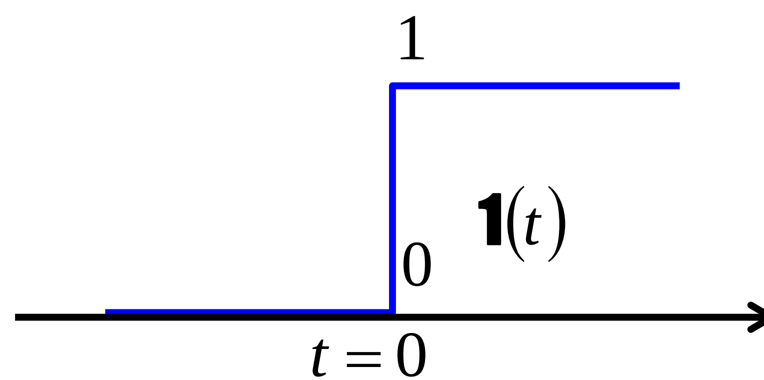
Fourier transform pairs



$$x(t)=\begin{cases} 0, & t<0 \\ e^{-t}, & t\geq 0 \end{cases} = \mathbf{1}(t)e^{-t}$$

$$X(\omega)=\int_0^{+\infty} e^{-t}e^{-j\omega t}\,dt=\int_0^{+\infty} e^{-(1+j\omega)t}\,dt$$

$$X(\omega)=\frac{1}{1+j\omega}$$



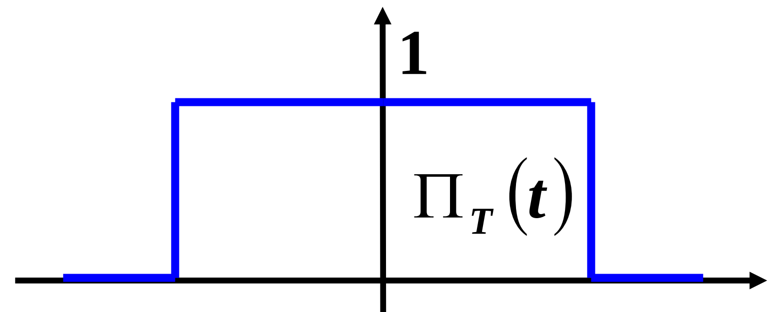
Unit step

FOURIER TRANSFORM

$$\mathbf{1}(t)e^{-t} \leftrightarrow \frac{1}{1+j\omega}$$

Fourier transform pairs

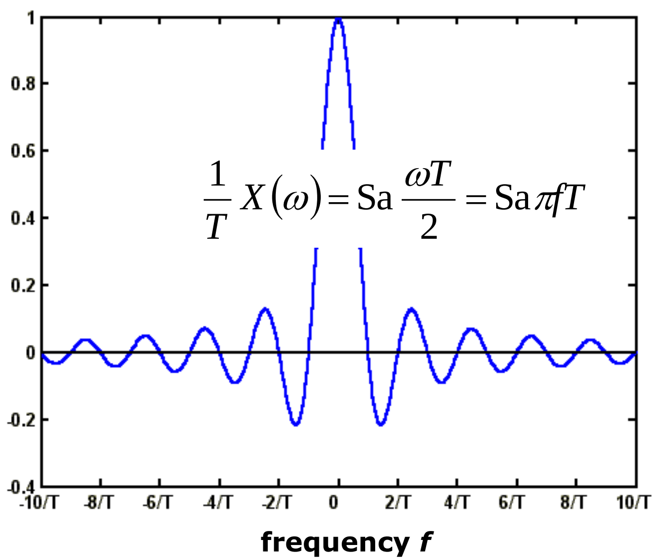
Rectangular pulse



$$x(t)=\begin{cases}0, & |t|>T/2\\ 1, & |t|\le T/2\end{cases}=\Pi_T(t)$$

$$X(\omega)=\int_{-T/2}^{+T/2}e^{-j\omega t}dt=T\mathrm{Sa}\frac{\omega T}{2}$$

$$\mathrm{Sa}(x)=\sin(x)/x$$ ***Sampling function***



FOURIER TRANSFORM

$$\Pi_T(t)\leftrightarrow T\mathrm{Sa}\frac{\omega T}{2}$$

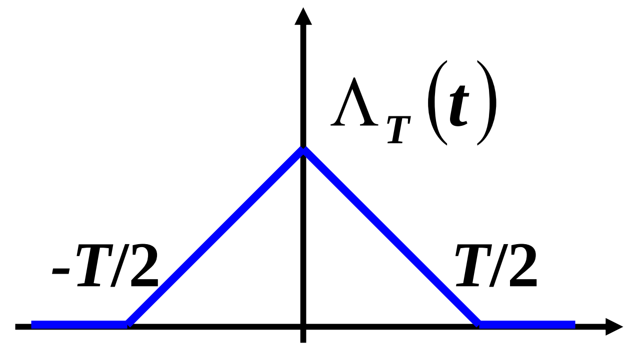
Fourier transform pairs

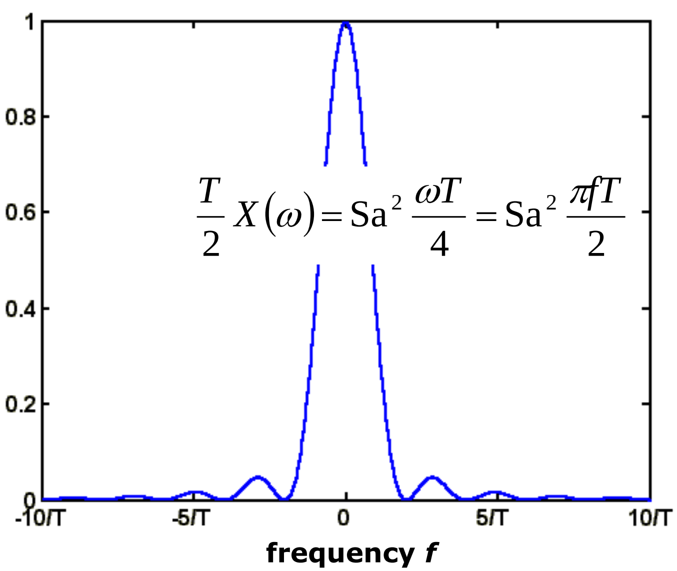
Traiangle pulse

$$x(t)=\begin{cases} 0, & |t|>T/2 \\ 1-2|t|/T, & |t|\le T/2 \end{cases} = \Lambda_T(t)$$

$$X(\omega)=\int_{-T/2}^{+T/2}\Lambda_T(t)=\frac{1}{2}T\,\mathrm{Sa}^2\frac{\omega T}{4}$$

$$\mathrm{Sa}^2(x)=\sin^2(x)/x^2$$





FOURIER TRANSFORM

$$\Lambda_T(t)\leftrightarrow\frac{1}{2}T\,\mathrm{Sa}^2\frac{\omega T}{4}$$

Summary

The response of Time-Invariant Linear System (TILS) to a composite input is a sum of responses to individual input components.

The exponential signal is an invariant to Linear Time-invariant Systems (TILS) with respect to a transfer function.

Fourier series and Fourier transform represent a signal through its spectrum which is a linear combination of exponential signals.

Fourier series is suited for periodic signals while Fourier transform is suited for unperiodic signals.

Trigonometric Fourier series

Fourier series coefficients of a real signal meet the Hermite symmetry:

$$X_n = X_{-n}^*$$

Exponential form of the Fourier series coefficient:

$$X_n = |X_n| e^{j\varphi_n}$$

Amplitude of exponential Fourier series coeff's is an even function:

$$|X_n| = |X_{-n}|$$

Phase of exponential Fourier series coeff's is an odd function:

$$\varphi_n = -\varphi_{-n}$$

Trigonometric Fourier series

Fourier series coefficients of a real signal meet the Hermite symmetry:

$$X_n = X_{-n}^* \qquad |X_n| = |X_{-n}| \qquad \varphi_n = -\varphi_{-n}$$

Exponential form of the Fourier series coefficient:

$$X_n = |X_n| e^{j\varphi_n}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} |X_n| \exp j\left(n\omega_0 t + \varphi_n\right) =$$

$$= X_0 + 2\sum_{n=1}^{\infty} |X_n| \cos\left(n\omega_0 t + \varphi_n\right)$$

The trigonometric Fourier series represents a signal as a compostion of real harmonic signals with different amplitudes and initial phases.

Trigonometric Fourier series

$$x(t) = X_0 + 2\sum_{n=1}^{\infty} |X_n| \cos(n\omega_0 t + \theta_n)$$

$$x(t) = \underbrace{X_0}_{c} + \sum_{n=1}^{\infty} \Big(\underbrace{2|X_n| \cos\theta_n}_{a_n} \cos n\omega_0 t + \underbrace{(-2|X_n| \sin\theta_n)}_{b_n} \sin n\omega_0 t\Big)$$

Prove

$$X_n = |X_n| e^{j\varphi_n} = |X_n| \cos\varphi_n + j|X_n| \sin\varphi_n$$

$$X_n = \tfrac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt = \tfrac{1}{T}\int_T x(t) \cos n\omega_0 t dt + j\tfrac{-1}{T}\int_T x(t) \sin n\omega_0 t dt$$

Trigonometric Fourier series:

$$x(t) = c + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$c = \frac{1}{T}\int_0^T x(t) dt$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t dt \qquad b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_0 t dt$$

Prove